

PIECEWISE PRINCIPAL COACTIONS OF CO-COMMUTATIVE HOPF ALGEBRAS

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ABSTRACT. Principal comodule algebras can be thought of as objects representing principal bundles in non-commutative geometry. A crucial component of a principal comodule algebra is a strong connection map. For some applications it suffices to prove that such a map exists, but for others, such as computing the associated bundle projectors or Chern-Galois characters, an explicit formula for a strong connection is necessary. It has been known for some time how to construct a strong connection map on a multi-pullback comodule algebra from strong connections on multi-pullback components, but the known explicit general formula is unwieldy. In this paper we derive a much easier to use strong connection formula, which is not, however, completely general, but is applicable only in the case when a Hopf algebra is co-commutative. Because certain linear splittings of projections in multi-pullback comodule algebras play a crucial role in our construction, we also devote a significant part of the paper to the problem of existence and explicit formulas for such splittings. Finally we show example application of our work.

1. INTRODUCTION

Let H be a Hopf algebra (with bijective antipode), interpreted as a Peter-Weyl algebra of functions on a quantum group. Principal H -comodule algebras can be loosely viewed as the algebras of appropriate classes of functions on (non-commutative) principal bundles ([2] makes the relationship explicit in the classical case). A crucial ingredient in the definition of principal comodule algebra is a so called strong connection map. For some applications it suffices to prove that a strong connection map exists, for instance when proving principality of a comodule algebra (see e.g. [18]). Other applications (see e.g. [3], [15], [16], [19]), such as computing the associated bundle projector or Chern-Galois character [5], call for an explicit formula for this map.

Piecewise principal comodule algebras [10], [13] is an interesting class of principal comodule algebras for which a fair amount of examples recently appeared in the literature (see e.g. [1], [4], [6], [8], [14], [15], [17], [18], [19]). They can be understood as being glued (constructed as a multi-pullback) from simpler parts which are principal. In [13] (cf. the generalization in [19]) it was proven that piecewise principal comodule algebras are, in fact, principal. The paper contains a derivation of the explicit formula for a strong connection on a pullback of two principal extensions from the “local” strong connections on pullback components and an appropriate choice of splittings of the gluing maps. If the piecewise comodule algebra is a multipullback one can present this multipullback as an iterated pullback, and then iterate the formula. Unfortunately, in practice, already the second iteration of the formula from [13] becomes overly complicated.

In the paper we derive, under the assumption of the co-commutativity of the Hopf algebra, a much simpler strong connection formula (which does not need to be iterated, nor requires putting the multipullback in the iterated form — the latter being complicated and error prone

by itself). While the assumption of co-commutativity limits severely the applicability of the formula, it is worth pointing out that many of the known piecewise principal comodule algebras, such as those considered in [17], [19], [14], [1], [15] and [18] are either $C(\mathbb{Z}_n)$ or $\mathcal{O}(U(1))$ -comodule algebras, hence our result could have been used to compute strong connections for these examples. The strong connection formula presented in this paper was inspired (very loosely) by the proof of [22, Theorem 3.3.2].

The plan of the paper is as follows: Section 2 contains some preliminaries about principal comodule algebras and piecewise principality. In Section 3 we present the explicit formula for a strong connection, and prove that it is indeed a strong connection, as long as the Hopf algebra is co-commutative. Because the strong connection formula uses the colinear and unital splittings of projections onto pieces, we devote the Section 4 to the presentation of the explicit procedure for constructing such splittings from the appropriate splittings of the gluing maps. Note that Theorem 7 can be viewed as the strengthening of [9, Proposition 9] (cf. [20, Theorem 7]) — instead of merely showing that, for each element in the multipullback component, there exists an element in the multipullback projected to this element we explicitly construct the whole (co-)linear and unital splitting.

As some of the splittings of gluing maps used in the construction of the splitting from Theorem 7 are required to have fairly non-obvious properties, the Section 5 is devoted to showing when such a splittings are guaranteed to exist, as well as to their semi-explicit constructions. Lemma 8, which links the existence of certain partitions of a vector space generated by a collection of vector subspaces to the distributivity of the lattice generated by those subspaces, is crucial for the results in this section.

Finally, in Section 6, we derive a formula for a strong connection on a non-commutative sphere $S^2_{\mathbb{R}^T}$ introduced in [18] as a quantum \mathbb{Z}_2 -principal bundle. To this end, and to provide comparison, we use two methods — the one from [13] and the one introduced in this paper.

2. PRELIMINARIES

2.1. Hopf algebra and comodule-related notation. We work over a fixed ground field \mathbb{K} and, unless stated otherwise, all vector spaces are understood to be \mathbb{K} -vector spaces and the unadorned tensor product is understood to be the algebraic tensor product over \mathbb{K} . The comultiplication, counit and the antipode of a Hopf algebra H are denoted by Δ , ϵ and S , respectively. Let P be a right comodule algebra. We denote by $\Delta_P : P \rightarrow P \otimes H$ the right H -coaction on P , and by

$$P^{\text{co}H} := \{p \in P \mid \Delta_P(p) = p \otimes 1_H\}$$

the subalgebra of coaction invariant elements. Instead of writing Δ 's and Δ_P 's we usually employ the Heynemann-Sweedler notation with the summation symbol suppressed, e.g.,:

$$\Delta(h) =: h_{(1)} \otimes h_{(2)}, \quad \Delta_P(p) =: p_{(0)} \otimes p_{(1)}$$

2.2. Principal comodule algebras. Let H be a Hopf algebra with bijective antipode, and let P be a right H -comodule algebra. Then P is a principal comodule algebra if and only if there exists a linear map

$$\ell : H \rightarrow P \otimes P, \quad \ell(h) =: \ell(h)^{(1)} \otimes \ell(h)^{(2)}$$

(note the Sweedler-like notation with summation sign suppressed) satisfying the following conditions:

$$\begin{aligned}
 (1a) \quad & \ell(1_H) = 1_P \otimes 1_P \\
 (1b) \quad & \ell(h)^{\langle 1 \rangle} \ell(h)^{\langle 2 \rangle} = \epsilon(h), \\
 (1c) \quad & \ell(h_{(1)})^{\langle 1 \rangle} \otimes \ell(h_{(1)})^{\langle 2 \rangle} \otimes h_{(2)} = \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}_{(1)}, \\
 (1d) \quad & S(h_{(1)}) \otimes \ell(h_{(2)})^{\langle 1 \rangle} \otimes \ell(h_{(2)})^{\langle 2 \rangle} = \ell(h)^{\langle 1 \rangle}_{(1)} \otimes \ell(h)^{\langle 1 \rangle}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}.
 \end{aligned}$$

Such a map, if it exists, is called a *strong connection* on P [11, 7, 5]. Strong connections are usually non-unique.

2.3. Multi-pullbacks of algebras. Let J be a finite set, and let

$$(2) \quad \{\pi_j^i : A_i \longrightarrow A_{ij} = A_{ji}\}_{i,j \in J, i \neq j}$$

be a family of algebra homomorphisms to which we will occasionally refer as “gluing maps”.

DEFINITION 1 ([9, 23]). *The multi-pullback algebra A^π of a family (2) of algebra homomorphisms is defined as*

$$A^\pi := \left\{ (a_i)_i \in \prod_{i \in J} A_i \mid \pi_j^i(a_i) = \pi_i^j(a_j), \forall i, j \in J, i \neq j \right\}.$$

DEFINITION 2 ([20]). *A family (2) of algebra homomorphisms is called distributive if and only if all of them are surjective and their kernels generate a distributive lattice of ideals.*

Let $(\pi_j^i : A_i \rightarrow A_{ij})_{i,j}$ be a family of surjective algebra homomorphisms. For any distinct i, j, k we put $A_{jk}^i := A_i / (\ker \pi_j^i + \ker \pi_k^i)$ and take $[\cdot]_{jk}^i : A_i \rightarrow A_{jk}^i$ to be the canonical surjections. Next, we introduce the family of maps

$$(3) \quad \pi_k^{ij} : A_{jk}^i \longrightarrow A_{ij} / \pi_j^i(\ker \pi_k^i), \quad [a_i]_{jk}^i \longmapsto \pi_j^i(a_i) + \pi_j^i(\ker \pi_k^i).$$

They are isomorphisms when π_j^i 's are epimorphisms.

DEFINITION 3. *We say [9, in Proposition 9] that a family $(\pi_j^i : A_i \rightarrow A_{ij})_{i,j}$ of algebra epimorphisms satisfies the cocycle condition if and only if, for all distinct $i, j, k \in J$,*

$$\begin{aligned}
 (1) \quad & \pi_j^i(\ker \pi_k^i) = \pi_i^j(\ker \pi_k^j), \\
 (2) \quad & \text{the isomorphisms } \phi_k^{ij} := (\pi_k^{ij})^{-1} \circ \pi_k^{ji} : A_{ik}^j \rightarrow A_{jk}^i \text{ satisfy } \phi_j^{ik} = \phi_k^{ij} \circ \phi_i^{jk}.
 \end{aligned}$$

Observe that, for all distinct $i, j, k \in J$ and any $a_i \in A_i, a_j \in A_j$,

$$(4) \quad [a_i]_{jk}^i = \phi_k^{ij}([a_j]_{ik}^j) \Leftrightarrow \pi_k^{ji}([a_j]_{ik}^j) = \pi_k^{ij}([a_i]_{jk}^i) \Leftrightarrow \pi_j^i(a_i) - \pi_i^j(a_j) \in \pi_j^i(\ker \pi_k^i).$$

One can prove ([9], cf. [20], see also Theorem 7 in this paper) that the cocycle condition guarantees that all projections on components of a multipullback are surjective (in fact all projections on submultipullbacks are surjective, but we will not make use of that fact).

2.4. Piecewise principal comodule algebras.

DEFINITION 4. (cf. [13, Definition 3.7]) A family of surjective algebra homomorphisms $\{\pi_i : P \rightarrow P_i\}_{i \in \{1, \dots, N\}}$ is called a covering [13] if and only if

- (1) $\bigcap_{i \in \{1, \dots, N\}} \ker \pi_i = \{0\}$,
- (2) The family of ideals $(\ker \pi_i)_{i \in \{1, \dots, N\}}$ generates a distributive lattice with $+$ and \cap as meet and join respectively.

Piecewise principal comodule algebras generalize the notion of (algebras of functions on) classical spaces which are locally principal, but with respect to closed instead of open coverings — hence the use of the term “piecewise” instead of “locally”.

DEFINITION 5. (see [13, Definition 3.8]) An H -comodule algebra P is called piecewise principal if there exists a family $\{\pi_i : P \rightarrow P_i\}_{i \in J}$ of surjective H -comodule algebra morphisms such that:

- (1) The restrictions $\pi_i|_{P^{coH}} : P^{coH} \rightarrow P_i^{coH}$ form a covering.
- (2) The P_i ’s are principal H -comodule algebras.

By [13, Theorem 3.3] a piecewise principal comodule algebra is principal. Note that any piecewise principal comodule algebra can be presented as a multipullback comodule algebra with the gluing maps being comodule algebra morphisms [10].

3. STRONG CONNECTION FORMULA

In this section we present an explicit (and arguably simple) expression for a strong connection on a piecewise principal H -comodule algebra where H is a co-commutative Hopf algebra. Regretfully, the co-commutativity assumption is used crucially in the proof of the correctness of the formula, and so we have little hopes of generalizing further the method which led to the derivation of this strong connection formula.

THEOREM 6. Let H be a cocommutative Hopf algebra. Let $\{\pi_i : P \rightarrow P_i\}_{i \in \{0, \dots, n\}}$ be a piecewise principal H -comodule algebra, and let $\{\ell_i : H \rightarrow P_i \otimes P_i\}_{i \in \{0, \dots, n\}}$ denote a family of strong connections on P_i ’s. For any $i \in \{0, \dots, n\}$, let V_i be an H sub-comodule of P_i such that $\ell_i(H) \subseteq V_i \otimes V_i$ and let $\alpha_i : V_i \rightarrow P$ be a unital, colinear splitting of π_i , i.e., $\pi_i \circ \alpha_i = \text{id}_{V_i}$. For brevity, denote for $i \in \{0, \dots, n\}$, $h \in H$

$$\begin{aligned} \theta_i(h) &:= \epsilon(h) - \alpha_i(\ell_i(h)^{(1)})\alpha_i(\ell_i(h)^{(2)}), \\ T_i(h) &:= \theta_i(h_{(1)})\theta_{i+1}(h_{(2)}) \cdots \theta_n(h_{(n-i+1)}), \quad T_{n+1}(h) := \epsilon(h). \end{aligned}$$

Then the linear map $\ell : H \rightarrow P \otimes P$ defined for all $h \in H$ by the formula

$$\ell(h) = \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \otimes \alpha_i(\ell_i(h_{(1)})^{(2)}) T_{i+1}(h_{(2)})$$

is a strong connection on P .

Proof. Note that any co-commutative Hopf algebra has bijective (in fact involutive) antipode. We need to prove that the map ℓ defined in the Theorem, satisfies all the properties (1).

First note that, by the colinearity of α_j ’s, colinear properties (1d), (1c) of ℓ_j ’s and the co-commutativity of H we have, that $\alpha_j(\ell_j(h)^{(1)})\alpha_j(\ell_j(h)^{(2)})$ is a coaction invariant element of P

for any $j \in \{0, \dots, n\}$ and $h \in H$, and hence also $T_i(h)$ is a coaction invariant element of P for any $i \in \{0, \dots, n+1\}$ and $h \in H$:

$$\begin{aligned} \rho^H(\alpha_j(\ell_j(h)^{(1)})\alpha_j(\ell_j(h)^{(2)})) &= \alpha_j(\ell_j(h)^{(1)})_{(0)}\alpha_j(\ell_j(h)^{(2)})_{(0)} \otimes \alpha_j(\ell_j(h)^{(1)})_{(1)}\alpha_j(\ell_j(h)^{(2)})_{(1)} \\ &= \alpha_j(\ell_j(h_{(2)})^{(1)})\alpha_j(\ell_j(h_{(2)})^{(2)}) \otimes S(h_{(1)})h_{(3)} \\ &= \alpha_j(\ell_j(h_{(1)})^{(1)})\alpha_j(\ell_j(h_{(1)})^{(2)}) \otimes S(h_{(2)})h_{(3)} \\ &= \alpha_j(\ell_j(h)^{(1)})\alpha_j(\ell_j(h)^{(2)}) \otimes 1. \end{aligned}$$

In the penultimate equality we used co-commutativity of H to swap Sweedler indices $_{(1)}$ and $_{(2)}$ to be able to use the antipode property. In order to prove that ℓ is left colinear (Equation (1d)) we use the left colinearity of ℓ_i 's, the colinearity of α_i 's and the coaction invariance of the $T_i(h)$'s:

$$\begin{aligned} &\ell(h)^{(1)}_{(1)} \otimes \ell(h)^{(1)}_{(0)} \otimes \ell(h)^{(2)} \\ &= \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)})_{(1)} \otimes \alpha_i(\ell_i(h_{(1)})^{(1)})_{(0)} \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})T_{i+1}(h_{(2)}) \\ &= \sum_{i=0}^n \ell_i(h_{(1)})^{(1)}_{(1)} \otimes \alpha_i(\ell_i(h_{(1)})^{(1)})_{(0)} \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})T_{i+1}(h_{(2)}) \\ &= \sum_{i=0}^n S(h_{(1)}) \otimes \alpha_i(\ell_i(h_{(2)})^{(1)}) \otimes \alpha_i(\ell_i(h_{(2)})^{(2)})T_{i+1}(h_{(3)}) \\ &= S(h_{(1)}) \otimes \ell(h_{(2)})^{(1)} \otimes \ell(h_{(2)})^{(2)}. \end{aligned}$$

The right colinearity (Equation (1c)) of ℓ follows from the H -coaction invariance of $T_i(h)$'s, the right colinearity of ℓ_i 's, the colinearity of α_i 's, and the co-commutativity of H :

$$\begin{aligned} &\ell(h)^{(1)} \otimes \ell(h)^{(2)}_{(0)} \otimes \ell(h)^{(2)}_{(1)} \\ &= \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})_{(0)}T_{i+1}(h_{(2)}) \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})_{(1)} \\ &= \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})_{(0)}T_{i+1}(h_{(2)}) \otimes \ell_i(h_{(1)})^{(2)}_{(1)} \\ &= \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})T_{i+1}(h_{(3)}) \otimes h_{(2)} \\ &= \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \otimes \alpha_i(\ell_i(h_{(1)})^{(2)})T_{i+1}(h_{(2)}) \otimes h_{(3)} \\ &= \ell(h_{(1)})^{(1)} \otimes \ell(h_{(1)})^{(2)} \otimes h_{(3)}. \end{aligned}$$

Here, in the penultimate inequality we used the co-commutativity of H exchanging Sweedler indices $_{(2)}$ and $_{(3)}$.

In order to prove that ℓ is unital (Equation (1a)), note first that for any $i \in \{0, \dots, n\}$

$$\theta_i(1) = \epsilon(1) - \alpha_i(\ell_i(1)^{(1)})\alpha_i(\ell_i(1)^{(2)}) = 1 - \alpha_i(1)\alpha_i(1) = 1 - 1 = 0,$$

because ϵ , all ℓ_i 's and all α_i 's are unital. It follows that $T_i(1) = 0$ for all $i \in \{0, \dots, n\}$, and $T_{n+1} = \epsilon$ by definition, hence

$$\ell(1) = \sum_{i=0}^n \alpha_i(\ell_i(1)^{(1)}) \otimes \alpha_i(\ell_i(1)^{(2)}) T_{i+1}(1) = \alpha_n(\ell_n(1)^{(1)}) \otimes \alpha_n(\ell_n(1)^{(2)}) T_{n+1}(1) = 1 \otimes 1,$$

where we used again the unitality of α_n and ℓ_n

Note now that for all $i \in \{0, \dots, n\}$, and $h \in H$

$$\begin{aligned} T_i(h) &= \epsilon(h_{(1)}) T_{i+1}(h_{(2)}) - \alpha_i(\ell_i(h_{(1)})^{(1)}) \alpha_i(\ell_i(h_{(1)})^{(2)}) T_{i+1}(h_{(2)}) \\ &= T_{i+1}(h) - \alpha_i(\ell_i(h_{(1)})^{(1)}) \alpha_i(\ell_i(h_{(1)})^{(2)}) T_{i+1}(h_{(2)}). \end{aligned}$$

By applying this formula to $T_0(h)$ and keeping to expand with it the leftmost summand of the resulting expansion we obtain easily:

$$(5) \quad T_0(h) = \epsilon(h) - \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \alpha_i(\ell_i(h_{(1)})^{(2)}) T_{i+1}(h_{(2)}).$$

On the other hand, for all $h \in H$ and $i \in \{0, \dots, n\}$, as α_i is the splitting of π_i it follows that:

$$\begin{aligned} \pi_i(\theta_i(h)) &= \epsilon(h) - \pi_i(\alpha_i(\ell_i(h)^{(1)})) \pi_i(\alpha_i(\ell_i(h)^{(2)})) \\ &= \epsilon(h) - \ell_i(h)^{(1)} \ell_i(h)^{(2)} \\ &= \epsilon(h) - \epsilon(h) \\ &= 0. \end{aligned}$$

Hence

$$\pi_i(T_j(h)) = 0, \quad \text{for all } i \geq j, \ i \in \{0, \dots, n\}, \ h \in H.$$

In particular, $\pi_i(T_0(h)) = 0$ for all $i \in \{0, \dots, n\}$ and $h \in H$. It follows that $T_0(h) = 0$ for all $h \in H$ because $\bigcap_{i=0}^n \ker \pi_i = \{0\}$, as $\{\pi_i : P \rightarrow P_i\}_{i \in \{0, \dots, n\}}$ is a covering. The last fact is an immediate consequence of [13, Theorem 3.3] and [13, Corollary 3.7].

Combining this with the Equation (5) we obtain that for all $h \in H$

$$\ell(h)^{(1)} \ell(h)^{(2)} = \sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{(1)}) \alpha_i(\ell_i(h_{(1)})^{(2)}) T_{i+1}(h_{(2)}) = \epsilon(h),$$

i.e., ℓ satisfies Equation (1b) as needed. \square

The expression for a strong connection provided in the above theorem requires the unital and colinear splittings of projections π_i to be given. The existence of such a splittings is guaranteed by the [13, Lemma 3.1] and [13, Theorem 3.3], but the mere existence does not suffice for someone desirous of finding the explicit formula. The proof of [13, Lemma 3.1] involves constructing a unital and colinear splitting of surjective comodule algebra map π from a unital and linear splitting of restriction of π to the subalgebra of coaction invariant elements (which always exists) utilizing the strong connection. Hence, we cannot use even the slight simplification provided by the proof of [13, Lemma 3.1].

In practice, we expect that in many simpler cases, the appropriate splittings will not be difficult to guess. However, for our result to be more widely applicable in practice, we will examine the explicit construction of colinear and unital splittings of multipullback comodule algebra projections on components which does not assume the existence of a strong connection on a multipullback comodule algebra (recall that a piecewise principal comodule algebra can always be presented as a multipullback).

4. COLINEAR SPLITTINGS OF PIECEWISE PRINCIPAL COMODULE ALGEBRAS

The result presented in this section allows to explicitly construct linear (colinear when appropriate) and unital splittings of projections on components of a multipullback (comodule) algebra.

THEOREM 7. *Suppose that a family (2) is distributive and satisfies the cocycle condition. Moreover suppose that there exists two families $\alpha_j^i, \beta_j^i : A_{ij} \rightarrow A_i$, $i, j \in J$, $j \neq i$ of linear (colinear) splittings of π_j^i 's such that all β_j^i 's are unital and for all distinct $i, j, k \in J$ we have*

$$(6) \quad \alpha_j^i(\pi_j^i(\ker \pi_k^i)) \subseteq \ker \pi_k^i.$$

Let $i \in J$, let $|J| = n + 1$ and let $\kappa : \{0, \dots, n\} \rightarrow J$ be a bijection such that $\kappa_0 = i$, where we denote $\kappa_j := \kappa(j)$ to ease the notation. Then a unital and linear (colinear) splitting $\alpha_i : A_i \rightarrow A^\pi$ of $\pi_i : A^\pi \rightarrow A_i$ can be given explicitly, for any $a \in A_i$ as $\alpha_i(a) := (a_j)_{j \in J}$, where $a_i := a$ and $a_{\kappa_{m+1}} := a_{\kappa_{m+1}}^m$ for any $0 \leq m < n$. The collections $\{a_{\kappa_{m+1}}^k\}_{0 \leq k \leq m} \subseteq A_{\kappa_{m+1}}$, for $0 \leq m < n$ are defined by the following inductive formula:

$$(7) \quad \begin{aligned} a_{\kappa_{m+1}}^0 &:= \beta_{\kappa_0}^{\kappa_{m+1}}(\pi_{\kappa_{m+1}}^{\kappa_0}(a_{\kappa_0})), \\ a_{\kappa_{m+1}}^{k+1} &:= a_{\kappa_{m+1}}^k - \alpha_{\kappa_{k+1}}^{\kappa_{m+1}}(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}})) \end{aligned}$$

for $0 \leq k < m$.

Proof. It is clear that because all the maps involved in the definition of α_i are unital and linear (colinear if need be) then also α_i is linear (resp. colinear). The proof of unitality is slightly more subtle and it requires a simple induction. Pick some bijection $\kappa : \{0, \dots, n\} \rightarrow J$ where $\kappa_0 = i$. Define $(a_j)_{j \in J} := \alpha_i(1)$. We need to show that $a_j = 1$ for all $j \in J$. Indeed, $a_{\kappa_0} = a_i = 1$ by definition. Suppose we have proven that $a_j = 1$ for all $0 \leq j \leq m < n$. Then using the Equation (7) we get $a_{\kappa_{m+1}}^0 = \beta_{\kappa_0}^{\kappa_{m+1}}(\pi_{\kappa_{m+1}}^{\kappa_0}(a_{\kappa_0})) = \beta_{\kappa_0}^{\kappa_{m+1}}(\pi_{\kappa_{m+1}}^{\kappa_0}(1)) = 1$ as both $\pi_{\kappa_{m+1}}^{\kappa_0}$ and $\beta_{\kappa_0}^{\kappa_{m+1}}$ are unital. Suppose now that we have proven that $a_{\kappa_{m+1}}^k = 1$ for all $0 \leq k < m$. Then, Equation (7) yields

$$\begin{aligned} a_{\kappa_{m+1}}^{k+1} &= a_{\kappa_{m+1}}^k - \alpha_{\kappa_{k+1}}^{\kappa_{m+1}}(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}})) \\ &= 1 - \alpha_{\kappa_{k+1}}^{\kappa_{m+1}}(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(1) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(1)) \\ &= 1 - \alpha_{\kappa_{k+1}}^{\kappa_{m+1}}(0) \\ &= 1. \end{aligned}$$

Now it remains to show that $\alpha_i(a) \in A^\pi$ for all $a \in A_i$. The inductive proof essentially follows the steps of the proof of [9, Proposition 9]. We will show that for any $0 \leq m \leq n$ we have

$$(8) \quad \pi_{\kappa_l}^{\kappa_j}(a_{\kappa_j}) = \pi_{\kappa_j}^{\kappa_l}(a_{\kappa_l}), \quad \text{for all } j, l \in \{0, \dots, m\}, j \neq l.$$

For $m = 0$ this condition is emptyly satisfied. Suppose we have proven the above condition for some m . In order to demonstrate it for $m + 1$, we prove by induction that for any $0 \leq k \leq m$, where $m < n$, we have

$$(9) \quad \pi_{\kappa_{m+1}}^{\kappa_j}(a_{\kappa_j}) = \pi_{\kappa_j}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k), \quad \text{for all } 0 \leq j \leq k.$$

If $k = 0$ then substituting the definition of $a_{\kappa_{m+1}}^0$ yields (as $\beta_{\kappa_0}^{\kappa_{m+1}}$ is a splitting of $\pi_{\kappa_0}^{\kappa_{m+1}}$):

$$\begin{aligned} \pi_{\kappa_0}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^0) &= \pi_{\kappa_0}^{\kappa_{m+1}}(\beta_{\kappa_0}^{\kappa_{m+1}}(\pi_{\kappa_{m+1}}^{\kappa_0}(a_{\kappa_0}))) \\ &= \pi_{\kappa_{m+1}}^{\kappa_0}(a_{\kappa_0}). \end{aligned}$$

Suppose now that we have proven Condition (9) for some $0 \leq k < m$. Pick any $0 \leq j \leq k$. Then by (inductively assumed) Condition (8) and Equation (4) we have

$$(10) \quad [a_{\kappa_j}]_{\kappa_{k+1}\kappa_{m+1}}^{\kappa_j} = \phi_{\kappa_{m+1}}^{\kappa_j\kappa_{k+1}} \left([a_{\kappa_{k+1}}]_{\kappa_j\kappa_{m+1}}^{\kappa_{k+1}} \right).$$

Then it follows that

$$\begin{aligned} [a_{\kappa_{m+1}}^k]_{\kappa_j\kappa_{k+1}}^{\kappa_{m+1}} &= \phi_{\kappa_{k+1}}^{\kappa_{m+1}\kappa_j} \left([a_{\kappa_j}]_{\kappa_{m+1}\kappa_{k+1}}^{\kappa_j} \right) && \{\text{By Condition (9) and Equation (4)}\} \\ &= \phi_{\kappa_{k+1}}^{\kappa_{m+1}\kappa_j} \left(\phi_{\kappa_{m+1}}^{\kappa_j\kappa_{k+1}} \left([a_{\kappa_{k+1}}]_{\kappa_j\kappa_{m+1}}^{\kappa_{k+1}} \right) \right) && \{\text{By Equation (10)}\} \\ &= \phi_{\kappa_j}^{\kappa_{m+1}\kappa_{k+1}} \left([a_{\kappa_{k+1}}]_{\kappa_j\kappa_{m+1}}^{\kappa_{k+1}} \right) && \{\text{By the cocycle condition}\}. \end{aligned}$$

This equality, again by Equation (4), is equivalent to the following condition:

$$\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}) \in \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(\ker \pi_{\kappa_{k+1}}^{\kappa_{m+1}}).$$

Because the above relation “is an element of” holds for an arbitrary $0 \leq j \leq k$ it implies immediately that

$$(11) \quad \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}) \in \bigcap_{0 \leq j \leq k} \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(\ker \pi_{\kappa_j}^{\kappa_{m+1}}).$$

Then

$$\begin{aligned} &\alpha_{\kappa_{k+1}}^{\kappa_{m+1}} \left(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}) \right) \\ \in &\alpha_{\kappa_{k+1}}^{\kappa_{m+1}} \left(\bigcap_{0 \leq j \leq k} \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(\ker \pi_{\kappa_j}^{\kappa_{m+1}}) \right) && \{\text{By Condition (11)}\} \\ \in &\bigcap_{0 \leq j \leq k} \alpha_{\kappa_{k+1}}^{\kappa_{m+1}} \left(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(\ker \pi_{\kappa_j}^{\kappa_{m+1}}) \right) && \{\text{By injectivity of } \alpha_{\kappa_{k+1}}^{\kappa_{m+1}}\} \\ \subseteq &\bigcap_{0 \leq j \leq k} \ker \pi_{\kappa_j}^{\kappa_{m+1}} && \{\text{By Equation (6)}\}, \end{aligned}$$

that is

$$(12) \quad \alpha_{\kappa_{k+1}}^{\kappa_{m+1}} \left(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}) \right) \in \bigcap_{0 \leq j \leq k} \ker \pi_{\kappa_j}^{\kappa_{m+1}}.$$

The above equation implies immediately, that for all $0 \leq l \leq k$

$$\begin{aligned} \pi_{\kappa_l}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^{k+1}) &= \pi_{\kappa_l}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_l}^{\kappa_{m+1}}(\alpha_{\kappa_{k+1}}^{\kappa_{m+1}}(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}))) \\ &= \pi_{\kappa_l}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) \\ &= \pi_{\kappa_{m+1}}^{\kappa_l}(a_{\kappa_l}), \end{aligned}$$

where, in the second equality we used the inductive assumption. Moreover, using the fact that $\alpha_{\kappa_{k+1}}^{\kappa_{m+1}}$ is a splitting of $\pi_{\kappa_{k+1}}^{\kappa_{m+1}}$ we obtain

$$\begin{aligned} \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^{k+1}) &= \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(\alpha_{\kappa_{k+1}}^{\kappa_{m+1}}(\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}))) \\ &= \pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - (\pi_{\kappa_{k+1}}^{\kappa_{m+1}}(a_{\kappa_{m+1}}^k) - \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}})) \\ &= \pi_{\kappa_{m+1}}^{\kappa_{k+1}}(a_{\kappa_{k+1}}), \end{aligned}$$

which ends the proof. \square

At this point, the skeptical reader might be excused for doubting the applicability of Theorem 7. Indeed, while the existence of unital and linear splittings β_j^i 's of π_j^i 's follows immediately from the surjectivity of π_j^i 's, and the existence of colinear splittings is assured (and assisted in explicit construction) by [13, Lemma 3.1] if all the A_i 's are principal comodule algebras, it is not clear how to find the linear splittings α_j^i satisfying Equation (6) nor that they exist at all in general case. Fortunately, the results from the next section, interesting in their own right, not only assure the existence of splittings α_j^i satisfying Equation (6) under no stronger assumptions than those of Theorem 7, but they also provide the method of their (semi)-explicit construction.

5. COLINEAR SPLITTINGS OF PRINCIPAL COMODULE ALGEBRAS

5.1. Partitions of sets. Let A be a set and let $A_i, i \in J$ be a fixed finite family of subsets of A . For any $\Gamma \in 2^J$ we denote for brevity:

$$(13) \quad A_\Gamma := \bigcap_{i \in \Gamma} A_i.$$

Obviously $A_{\Gamma_1} \cap A_{\Gamma_2} = A_{\Gamma_1 \cup \Gamma_2}$. Also $A_\emptyset = A$ by convention. It is easy to see that A_i 's generate a partition $\{B_\Gamma\}_{\Gamma \in 2^J}$ of A (i.e., all B_Γ 's are disjoint and $A = \bigcup_{\Gamma \in 2^J} B_\Gamma$) such that

$$(14) \quad A_\Gamma = \bigcup_{\Gamma' \in 2^J \mid \Gamma \subseteq \Gamma'} B_{\Gamma'}, \quad \text{for all } \Gamma \in 2^J.$$

Indeed, the partition can be described explicitly, for all $\Gamma \in 2^J$ by the formula:

$$B_\Gamma := \{x \in A \mid \forall i \in J. x \in A_i \equiv i \in \Gamma\}.$$

5.2. Partitions of vector spaces. Let now A be a vector space and let $A_i, i \in J$ be a fixed finite family of vector subspaces of A . A_Γ , for any $\Gamma \in 2^J$ is defined as in Equation (13). We want to define a linear counterpart of an associated partition $\{B_\Gamma\}_\Gamma$ defined above for sets. Similarly to plain sets, vector sub-spaces can be ordered by the set inclusion, and the resulting ordered set is a lattice, with subspace intersection ($V_1 \cap V_2$) serving as infimum and subspace sum ($V_1 + V_2$) playing the role of supremum. The problem is that this lattice is not, in general, distributive. It turns out that the assumption that the subspaces $A_i, i \in J$ generate a distributive lattice is pivotal for proving our desired result, stated immediately below:

LEMMA 8. *Let A be a linear vector space and let $A_i, i \in I$ be a finite family of vector subspaces of A generating a distributive lattice. A has a linear basis $\mathcal{B} = \bigcup_{\Gamma \in 2^I} \mathcal{B}_\Gamma$, where $\mathcal{B}_\Gamma \subseteq A_\Gamma$, $\Gamma \in 2^I$, such that subsets \mathcal{B}_Γ are all disjoint and satisfy the following property:*

$$(15) \quad A_\Gamma = \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma \supseteq \Gamma'} \mathcal{B}_{\Gamma'} \right)$$

for all $\Gamma \in 2^I$.

Proof. First fix a linear order \leq on 2^I subject to the condition

$$(16) \quad \Gamma_1 \supseteq \Gamma_2 \Rightarrow \Gamma_1 \leq \Gamma_2, \quad \text{for all } \Gamma_1, \Gamma_2 \in 2^I.$$

It is immediate that the minimal element in this order is I and maximal is \emptyset . Note the following property of \leq which will be used later:

$$(17) \quad \Gamma > \Gamma' \Rightarrow \Gamma \cup \Gamma' \supset \Gamma, \quad \text{for all } \Gamma, \Gamma' \in 2^I.$$

Indeed, assume $\Gamma > \Gamma'$. $\Gamma \cup \Gamma' \supseteq \Gamma$ always, so we need just to show that the equality leads to contradiction. Suppose that $\Gamma \cup \Gamma' = \Gamma$. This is equivalent to $\Gamma \supseteq \Gamma'$ which implies by Equation (16) that $\Gamma \leq \Gamma'$ contradicting the assumption $\Gamma > \Gamma'$.

The sets \mathcal{B}_Γ , $\Gamma \in 2^I$ can be generated inductively (with respect to \leq) as follows:

- (1) \mathcal{B}_I is some linear basis of A_I .
- (2) \mathcal{B}_Γ , for $\Gamma > I$, is chosen as a maximal subset of A_Γ such that $\bigcup_{\Gamma' \leq \Gamma} \mathcal{B}_{\Gamma'}$ is linearly independent.

It is immediate by construction of \mathcal{B}_Γ 's that $\mathcal{B} := \bigcup_{\Gamma \in 2^I} \mathcal{B}_\Gamma$ is a linear basis of A and that all \mathcal{B}_Γ 's are disjoint. Also by construction, $\mathcal{B}_{\Gamma'} \subseteq A_\Gamma$, $\Gamma \in 2^I$ whenever $\Gamma \subseteq \Gamma'$, which implies that half of Property (15) is trivially satisfied:

$$(18) \quad \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'} \right) \subseteq A_\Gamma$$

for all $\Gamma \in 2^I$. Finally, it is immediate that

$$(19) \quad A_\Gamma \subseteq \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma' \leq \Gamma} \mathcal{B}_{\Gamma'} \right).$$

We will prove the second half of Property (15) by induction on \leq :

- (1) I is minimal in 2^I with respect to \leq . Then by definition of \mathcal{B}_I we have

$$A_I = \text{Span}(\mathcal{B}_I) = \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma' \supseteq I} \mathcal{B}_{\Gamma'} \right).$$

- (2) Suppose we have proven Eq. (15) for all $\Gamma' < \Gamma$. For any $a \in A$, denote by $\{\alpha_\Gamma(a)\}_{\Gamma \in 2^I}$ the unique family of vectors such that $a = \sum_{\Gamma \in 2^I} \alpha_\Gamma(a)$ and that $\alpha_\Gamma(a) \in \text{Span}(\mathcal{B}_\Gamma)$ for all $\Gamma \in 2^I$ (they are unique because \mathcal{B} is a basis and \mathcal{B}_Γ 's are disjoint). By (19) $\alpha_{\Gamma'}(a) = 0$ whenever $a \in A_\Gamma$ and $\Gamma' > \Gamma$, i.e.,

$$(20) \quad a = \sum_{\Gamma' \in 2^I, \Gamma' \leq \Gamma} \alpha_{\Gamma'}(a), \quad \text{for all } a \in A_\Gamma.$$

Let $a \in A_\Gamma$. Define $v := a - \alpha_\Gamma(a)$. By Equation (20)

$$A_\Gamma \ni v = \sum_{\Gamma' \in 2^I, \Gamma' < \Gamma} \alpha_{\Gamma'}(a) \in \sum_{\Gamma' \in 2^I, \Gamma' < \Gamma} A_{\Gamma'},$$

hence

$$\begin{aligned}
 v &\in A_\Gamma \cap \left(\sum_{\Gamma' \in 2^I, \Gamma' < \Gamma} A_{\Gamma'} \right) \\
 &= \sum_{\Gamma' \in 2^I, \Gamma' < \Gamma} A_{\Gamma' \cup \Gamma} \quad (\text{By distributivity of lattice generated by } A_i\text{'s}) \\
 &\subseteq \sum_{\Gamma' \in 2^I, \Gamma' \supset \Gamma} A_{\Gamma'} \quad (\text{By Equation (17) } \Gamma \subset \Gamma \cup \Gamma' \text{ if } \Gamma' < \Gamma) \\
 &\subseteq \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma' \supset \Gamma} \mathcal{B}_{\Gamma'} \right) \quad (\text{By inductive assumption, as } \Gamma' < \Gamma \text{ if } \Gamma' \supset \Gamma).
 \end{aligned}$$

It follows that

$$a = \alpha_\Gamma(a) + v \in \text{Span}(\mathcal{B}_\Gamma) + \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma' \supset \Gamma} \mathcal{B}_{\Gamma'} \right) = \text{Span} \left(\bigcup_{\Gamma' \in 2^I, \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'} \right)$$

as needed. □

The following result is a common knowledge:

LEMMA 9. *Let $\pi : A \rightarrow B$ be a linear map, and let $\{A_i\}_{i \in I}$ be a finite family of vector subspaces of A . Assume that $\ker \pi \cap (\sum_{i \in I} A_i) = \sum_{i \in I} (\ker \pi \cap A_i)$. Then*

$$\pi \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} \pi(A_i).$$

LEMMA 10. *Let $\pi : A \rightarrow B$ be a linear epimorphism, and let $\{A_i\}_{i \in I}$ be a finite family of vector subspaces of A such that $\{A_i\}_{i \in I} \cup \{\ker \pi\}$ generates a distributive lattice of vector subspaces. Then there exists a linear splitting $\alpha : B \rightarrow A$ of π such that $\alpha(\pi(A_i)) \subseteq A_i$ for all $i \in I$.*

Proof. Let $\mathcal{B} := \bigcup_{\Gamma \in 2^I} \mathcal{B}_\Gamma$ be a linear basis of B satisfying conditions guaranteed by Lemma 8 with respect to the family $\{B_i\}_{i \in I}$, where $B_i := \pi(A_i)$. Note that Lemma 9 implies that B_i 's generate distributive lattice of ideals because A_i 's generate distributive lattice of ideals. For all $\Gamma \in 2^I$ such that \mathcal{B}_Γ is non-empty we define $\alpha(b)$ for all $b \in \mathcal{B}_\Gamma$, to be an arbitrary element of $\pi^{-1}(b) \cap A_\Gamma$. Note that $\pi^{-1}(b) \cap A_\Gamma$ is non-empty (so that this choice is possible) as $b \in B_\Gamma \neq \emptyset$, and, $B_\Gamma = \pi(A_\Gamma)$ by Lemma 9. The map $\alpha : B \rightarrow A$ thus obtained is clearly a linear splitting of π . For any $i \in I$ consider any $b \in B_i$. Then, by Lemma 8, $b \in \text{Span} \left(\bigcup_{\Gamma \in 2^I \mid i \in \Gamma} \mathcal{B}_\Gamma \right)$, and hence

$$\alpha(b) \in \sum_{\Gamma \in 2^I \mid i \in \Gamma} \sum_{b' \in \mathcal{B}_\Gamma} (\pi^{-1}(b') \cap A_\Gamma) \subseteq \sum_{\Gamma \in 2^I \mid i \in \Gamma} A_\Gamma \subseteq A_i.$$

□

Finally, we argue that we can generate a colinear splitting (with appropriate properties) from the linear one on the coaction invariant subalgebra:

LEMMA 11. *Let A be a principal H -comodule algebra, let $\pi : A \rightarrow B$ be an H -comodule algebra surjection, and let $\{A_i\}_{i \in I}$ be a finite family of ideals in A which are subcomodules, such that $\{A_i\}_{i \in I} \cup \{\ker \pi\}$ generates a distributive lattice. Define for all $i \in I$:*

$$A_i^{\text{co}H} := A_i \cap A^{\text{co}H}, \quad B_i := \pi(A_i), \quad B_i^{\text{co}H} := B^{\text{co}H} \cap B_i.$$

Suppose that there exists a linear map $\alpha^{\text{co}H} : B^{\text{co}H} \rightarrow A^{\text{co}H}$ such that

$$\pi \circ \alpha^{\text{co}H} = \text{id}_{B^{\text{co}H}}, \quad \alpha^{\text{co}H}(B_i^{\text{co}H}) \subseteq A_i^{\text{co}H}, \quad \text{for all } i \in I.$$

Let $\ell : H \rightarrow A \otimes A$ be a strong connection on A . Then the following formula:

$$\alpha : B \longrightarrow A, \quad b \longmapsto \alpha^{\text{co}H}(b_{(0)}\pi(\ell(b_{(1)})^{(1)}))\ell(b_{(1)})^{(2)}$$

defines a linear map satisfying

$$\pi \circ \alpha = \text{id}_B, \quad \alpha(B_i) \subseteq A_i, \quad \text{for all } i \in I.$$

Proof. The fact that α defined above is the colinear splitting of π follows immediately from the proof of [13, Lemma 3.1]. It remains to show that $\alpha(B_i) \subseteq A_i$ for all $i \in I$. Indeed, let $b \in B_i$. Because of the left colinearity of ℓ (Equation (1d)) it follows easily that $b_{(0)}\pi(\ell(b_{(1)})^{(1)}) \otimes \ell(b_{(1)})^{(2)} \in B^{\text{co}H} \otimes A$ (c.f. proof of [13, Lemma 3.1]), and because B_i is an ideal and also a right H -subcomodule, it follows also that $b_{(0)}\pi(\ell(b_{(1)})^{(1)}) \otimes \ell(b_{(1)})^{(2)} \in B_i \otimes A$, hence $b_{(0)}\pi(\ell(b_{(1)})^{(1)}) \otimes \ell(b_{(1)})^{(2)} \in B_i^{\text{co}H} \otimes A$. Therefore

$$\alpha(b) = \alpha^{\text{co}H}(b_{(0)}\pi(\ell(b_{(1)})^{(1)}))\ell(b_{(1)})^{(2)} \in \alpha^{\text{co}H}(B_i^{\text{co}H})A \subseteq A_i^{\text{co}H}A \subseteq A_i.$$

□

6. EXAMPLE

In [18] a new non-commutative real projective space $\mathbb{R}P_{\mathcal{T}}^2$ and a non-commutative sphere $S_{\mathbb{R}\mathcal{T}}^2$ were introduced, by defining $C(\mathbb{R}P_{\mathcal{T}}^2)$ and $C(S_{\mathbb{R}\mathcal{T}}^2)$ as a particular triple pullbacks of, respectively, three copies of the Toeplitz algebra \mathcal{T} and the tensor product $\mathcal{T} \otimes C(\mathbb{Z}_2)$. The algebra $C(S_{\mathbb{R}\mathcal{T}}^2)$ has a natural (component-wise) diagonal coaction of the Hopf algebra $C(\mathbb{Z}_2)$, and it was proven in [18] that the subspace of invariants of this coaction is isomorphic with $C(\mathbb{R}P_{\mathcal{T}}^2)$. Moreover, it was demonstrated that $C(S_{\mathbb{R}\mathcal{T}}^2)$ is a piecewise principal (hence principal) $C(\mathbb{Z}_2)$ -comodule algebra. However, the paper [18] does not present an explicit formula for a strong connection. Because $C(\mathbb{Z}_2)$ is co-commutative and $C(S_{\mathbb{R}\mathcal{T}}^2)$ is defined as a triple pullback algebra, our main result is applicable. In this section we will present the comparison of computations of a strong connection on $C(S_{\mathbb{R}\mathcal{T}}^2)$ using two methods: the first one uses the strong connection formula from [13] and the other one uses Theorem 6. The reader will see that, while application of the formula from [13] is trivial in case of double pullbacks, already for triple pullbacks the computations becomes fairly unmanageable. Also note that, in many cases, the values of strong connection formula on generators of the Hopf algebra are easily guessable, and then the values on arbitrary Hopf algebra elements can be computed using well known recursive formula. Here the Hopf algebra $C(\mathbb{Z}_2)$ has linear basis consisting of 1 and u , where u is the single generator such that $u^2 = 1$, so that it suffices to find the value of a strong connection on u without any need for recursion. However, guessing the value of a strong connection on u is nigh impossible.

We will start with recalling the definition of the comodule algebra $C(S_{\mathbb{R}\mathcal{T}}^2)$. Our presentation will be very brief (mostly lifted from [18]), though sufficient to understand what follows, and will hardly include any geometric intuitions behind $C(S_{\mathbb{R}\mathcal{T}}^2)$. Also, because the definition of

$C(\mathbb{R}P^2_{\mathcal{T}})$ is irrelevant for the strong connection computation, we omit it entirely. Therefore, the reader is recommended to read the full account from [18].

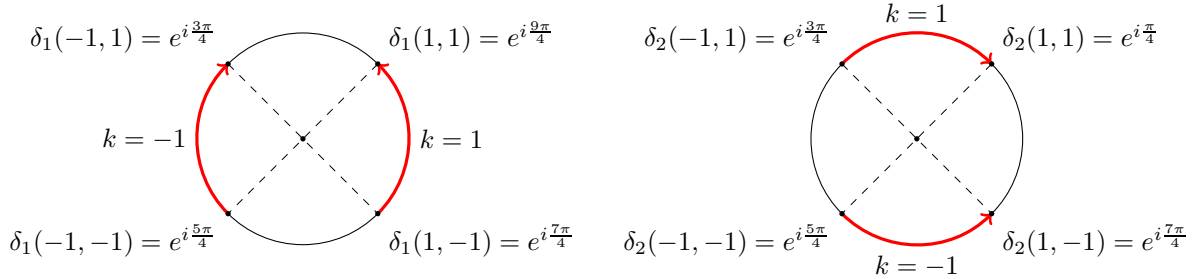
6.1. A pullback quantum sphere. We consider the Toeplitz algebra \mathcal{T} as the universal C^* -algebra generated by an isometry s , and the symbol map given by the assignment $\sigma: \mathcal{T} \ni s \mapsto \tilde{u} \in C(S^1)$, where \tilde{u} is the unitary function generating $C(S^1)$. The following two maps

$$(21) \quad \mathbb{Z}_2 \times I \ni (k, t) \xrightarrow{\delta_1} e^{i\pi(\frac{1}{4}kt + \frac{1}{2}k + \frac{3}{2})} \in S^1, \quad I \times \mathbb{Z}_2 \ni (t, k) \xrightarrow{\delta_2} e^{i\pi(-\frac{1}{4}kt - \frac{1}{2}k + 1)} \in S^1,$$

and their pullbacks

$$(22) \quad \delta_1^*: C(S^1) \longrightarrow C(\mathbb{Z}_2) \otimes C(I), \quad \delta_2^*: C(S^1) \longrightarrow C(I) \otimes C(\mathbb{Z}_2).$$

feature prominently in the definition of $C(S^2_{\mathbb{R}\mathcal{T}})$. We will denote for brevity $\sigma_i := \delta_i^* \circ \sigma$, $i = 1, 2$. The definitions of the δ_i 's seem completely arbitrary. In fact, as shown on the picture [18] below, each of these maps is meant as the parametrisation of two appropriate quarters of S^1 :



We view S^1 and I as \mathbb{Z}_2 -spaces via multiplication by ± 1 . Then $\mathbb{Z}_2 \times I$ and $I \times \mathbb{Z}_2$ are \mathbb{Z}_2 -spaces with the diagonal action. Accordingly, $C(I)$, $C(S^1)$, $C(\mathbb{Z}_2) \otimes C(I)$ and $C(I) \otimes C(\mathbb{Z}_2)$ are right $C(\mathbb{Z}_2)$ -comodule algebras with coactions given by the pullbacks of respective \mathbb{Z}_2 -actions. Denote by u the generator $C(\mathbb{Z}_2)$ given by $u(\pm 1) := \pm 1$. Then the assignment $s \mapsto s \otimes u$ makes \mathcal{T} a $C(\mathbb{Z}_2)$ -comodule algebra. (This coaction corresponds to the \mathbb{Z}_2 -action given by $\alpha_{-1}^{\mathcal{T}}(s) = -s$.) It is easy to verify that the maps δ_i , $i = 1, 2$, are \mathbb{Z}_2 -equivariant, so that their pullbacks δ_i^* 's are right $C(\mathbb{Z}_2)$ -comodule maps. Also, since the symbol map σ is a right $C(\mathbb{Z}_2)$ -comodule map, so are σ_i 's.

The construction of $C(S^2_{\mathbb{R}\mathcal{T}})$ can be seen as the quantum version of constructing the topological 2-sphere by assembling three pairs of squares to the boundary of a cube. In the quantum version the algebra $\mathcal{T} \otimes C(\mathbb{Z}_2)$ replaces the pair of squares. Explicitly, the algebra $C(S^2_{\mathbb{R}\mathcal{T}})$ is

defined in [18] to be the following triple pullback of three copies of $\mathcal{T} \otimes C(\mathbb{Z}_2)$:

$$\begin{array}{ccc}
 \mathcal{T}_0 \otimes C(\mathbb{Z}_2) & & \mathcal{T}_1 \otimes C(\mathbb{Z}_2) \\
 \sigma_1 \otimes \text{id} \downarrow & & \downarrow \sigma_1 \otimes \text{id} \\
 C(\mathbb{Z}_2) \otimes C(I) \otimes C(\mathbb{Z}_2) & \xleftarrow{\Phi_{01}} & C(\mathbb{Z}_2) \otimes C(I) \otimes C(\mathbb{Z}_2), \\
 \\
 \mathcal{T}_0 \otimes C(\mathbb{Z}_2) & & \mathcal{T}_2 \otimes C(\mathbb{Z}_2) \\
 \sigma_2 \otimes \text{id} \downarrow & & \downarrow \sigma_1 \otimes \text{id} \\
 C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2) & \xleftarrow{\Phi_{02}} & C(\mathbb{Z}_2) \otimes C(I) \otimes C(\mathbb{Z}_2), \\
 \\
 \mathcal{T}_1 \otimes C(\mathbb{Z}_2) & & \mathcal{T}_2 \otimes C(\mathbb{Z}_2) \\
 \sigma_2 \otimes \text{id} \downarrow & & \downarrow \sigma_2 \otimes \text{id} \\
 C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2) & \xleftarrow{\Phi_{12}} & C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2).
 \end{array}
 \tag{23}$$

where the isomorphisms Φ_{ij} are defined by the following formulas, for all $h, k \in C(\mathbb{Z}_2)$ and $p \in C(I)$:

$$\begin{aligned}
 \Phi_{01}(h \otimes p \otimes k) &:= k \otimes p \otimes h, \\
 \Phi_{02}(h \otimes p \otimes k) &:= p \otimes k \otimes h, \\
 \Phi_{12}(p \otimes h \otimes k) &:= p \otimes k \otimes h.
 \end{aligned}
 \tag{24}$$

We view the algebras $\mathcal{T} \otimes C(\mathbb{Z}_2)$, $C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2)$ and $C(\mathbb{Z}_2) \otimes C(I) \otimes C(\mathbb{Z}_2)$ as right $C(\mathbb{Z}_2)$ -comodules with the diagonal $C(\mathbb{Z}_2)$ -coaction. The coaction of $C(\mathbb{Z}_2)$ is defined on $C(S_{\mathbb{R}\mathcal{T}}^2)$ componentwise.

6.2. Construction of certain auxilliary elements. Both constructions of strong connections will require the existence of elements $\phi_1 \in \sigma_1^{-1}(u \otimes 1_{C(I)}) \subseteq \mathcal{T}$, $\phi_2 \in \sigma_2^{-1}(1_{C(I)} \otimes u) \subseteq \mathcal{T}$ with certain additional properties. These elements will play the crucial role in the construction of appropriate splittings required by both methods. More explicitly, we have the following:

LEMMA 12. *There exist elements $\phi_1, \phi_2 \in \mathcal{T}$ satisfying:*

$$\begin{aligned}
 (25a) \quad & \rho(\phi_1) = \phi_1 \otimes u, \quad \rho(\phi_2) = \phi_2 \otimes u, \\
 (25b) \quad & \sigma_1(\phi_1) = u \otimes 1_{C(I)}, \quad \sigma_2(\phi_1) = \iota_I \otimes 1_{C(\mathbb{Z}_2)}, \quad \sigma_2(\phi_2) = 1_{C(I)} \otimes u, \quad \sigma_1(\phi_2) = 1_{C(\mathbb{Z}_2)} \otimes \iota_I, \\
 (25c) \quad & (1 - \phi_2^2)(1 - \phi_1^2) \neq 0.
 \end{aligned}$$

where $\iota_I \in C(I)$ is an identity map $\iota_I(t) = t$ and $\rho : \mathcal{T} \rightarrow \mathcal{T} \otimes C(\mathbb{Z}_2)$ is a right coaction.

Proof. First we define auxiliary maps $\hat{\phi}_1, \hat{\phi}_2 \in C(S^1)$ by the formulae:

$$(26) \quad \hat{\phi}_1(e^{i\theta}) := \begin{cases} 2 - \frac{4}{\pi}\theta & \text{if } \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \\ -1 & \text{if } \theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \\ \frac{4}{\pi}\theta - 6 & \text{if } \theta \in [\frac{5\pi}{4}, \frac{7\pi}{4}] \\ 1 & \text{if } \theta \in [\frac{7\pi}{4}, \frac{9\pi}{4}] \end{cases}, \quad \hat{\phi}_2(e^{i\theta}) := \begin{cases} 1 & \text{if } \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \\ 4 - \frac{4}{\pi}\theta & \text{if } \theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \\ -1 & \text{if } \theta \in [\frac{5\pi}{4}, \frac{7\pi}{4}] \\ \frac{4}{\pi}\theta - 8 & \text{if } \theta \in [\frac{7\pi}{4}, \frac{9\pi}{4}] \end{cases}.$$

One immediately verifies that

$$(27) \quad \hat{\phi}_1, \hat{\phi}_2 : S^1 \longrightarrow [-1, 1], \quad \hat{\phi}_1(-z) = -\hat{\phi}_1(z), \quad \hat{\phi}_2(-z) = -\hat{\phi}_2(z).$$

(i.e., $\rho(\hat{\phi}_1) = \hat{\phi}_1 \otimes u$ and $\rho(\hat{\phi}_2) = \hat{\phi}_2 \otimes u$) and that

$$(28) \quad \hat{\phi}_1 \circ \delta_1 = u \otimes 1_{C(I)}, \quad \hat{\phi}_2 \circ \delta_2 = 1_{C(I)} \otimes u, \quad \hat{\phi}_1 \circ \delta_2 = \iota_I \otimes 1_{C(\mathbb{Z}_2)}, \quad \hat{\phi}_2 \circ \delta_1 = 1_{C(\mathbb{Z}_2)} \otimes \iota_I.$$

Using Equation (27) and the standard properties of comodules, one proves that because the symbol map σ is a surjective right $C(\mathbb{Z}_2)$ comodule map and u is grouplike, we can choose elements $\phi_i \in \mathcal{T}$, $i = 1, 2$, such that $\sigma(\phi_i) = \hat{\phi}_i$, and $\rho(\phi_i) = \phi_i \otimes u$, thus verifying the properties (25a). That thus chosen elements ϕ_1, ϕ_2 satisfy the properties (25b) follows immediately from Equations (28).

The last condition of the Lemma requires more work. Unfortunately, $\sigma((1 - \phi_2^2)(1 - \phi_1^2)) = (1 - \hat{\phi}_2^2)(1 - \hat{\phi}_1^2) = 0$, so we cannot prove that vector $(1 - \phi_2^2)(1 - \phi_1^2) \in \mathcal{T}$ is nonzero by considering the properties of its image in $C(S^1)$ under σ and we must work directly in \mathcal{T} . We will use the flexibility afforded by the fact that conditions (25a) and (25b) do not fix completely elements $\phi_i \in \mathcal{T}$. We will show that even if $(1 - \phi_2^2)(1 - \phi_1^2) = 0$ for our initial choice of ϕ_i 's, there exists a family $\{\phi_{2;t,n}\}_{t \in \mathbb{R}, n \in \mathbb{N}}$ of deformations of ϕ_2 such that the conditions (25a) and (25b) are still satisfied for all pairs $(\phi_1, \phi_{2;t,n})$ and there exist $n \in \mathbb{N}$ and $t \in \mathbb{R}$ such that $(1 - \phi_{2;t,n}^2)(1 - \phi_1^2) \neq 0$.

Let z be a partial isometry generating \mathcal{T} , and let $\rho : \mathcal{T} \rightarrow \mathcal{T} \otimes C(\mathbb{Z}_2)$ be a right $C(\mathbb{Z}_2)$ -coaction. Define, for all $n \in \mathbb{N}$, $t \in \mathbb{R}$

$$(29) \quad \phi_{2;t,n} := \phi_2 + tE_n, \quad \text{where} \quad E_n = z(z^*(z^*)^n - z^{n+2}(z^*)^{n+2}).$$

Because $\rho(z) = z \otimes u$, we have $\rho(\phi_{2;t,n}) = \phi_{2;t,n} \otimes u$ and because $\sigma(E_n) = 0$, we have $\sigma(\phi_{2;t,n}) = \hat{\phi}_2$, hence all of the conditions (25) are satisfied, and for all t and n we can use $\phi_{2;t,n}$ instead of ϕ_2 in the formula (51) defining a strong connection on $C(S_{\mathbb{R}\mathcal{T}}^2)$. Assume that $(1 - \phi_{2;t,n}^2)(1 - \phi_1^2) = 0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. We will show that this assumption leads to contradiction. Using eq. (29) elements $(1 - \phi_{2;t,n}^2)(1 - \phi_1^2)$ can be explicitly written as

$$(30) \quad (1 - \phi_2^2)(1 - \phi_1^2) - (E_n\phi_2 + \phi_2E_n)t - (E_n)^2(1 - \phi_1^2)t^2.$$

If $(1 - \phi_{2;t,n}^2)(1 - \phi_1^2) = 0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$ then the above polynomials in t are identically zero for all $n \in \mathbb{N}$, which implies in particular that coefficients at t^2 must be zero, i.e., that

$$(31) \quad E_n^2(1 - \phi_1^2) = 0, \quad \text{for all } n \in \mathbb{N}.$$

Consider now the faithful representation $R : \mathcal{T} \rightarrow \mathcal{H}$ of the Toeplitz algebra \mathcal{T} on a Hilbert space \mathcal{H} spanned by an orthonormal basis $|n\rangle$, $n \in \mathbb{N}$, where the partial isometry z is represented as a right shift, i.e., $R(z)|n\rangle = |n+1\rangle$ for all $n \in \mathbb{N}$. One easily proves that

$$(32) \quad R(E_n^2)|m\rangle = \delta_{m,n}|n+2\rangle, \quad \text{for all } m, n \in \mathbb{N}.$$

Eq. 31 implies that $R(E_n^2)R(1 - \phi_1^2)\Psi = 0$, for all $n \in \mathbb{N}$ and $\Psi \in \mathcal{H}$. But then it follows from eq. (32) that $R(1 - \phi_1^2)\Psi = 0$ for all $\Psi \in \mathcal{H}$, i.e., that $R(1 - \phi_1^2) = 0$. But R is faithful, hence $(1 - \phi_1^2) = 0$. On the other hand, $\sigma(1 - \phi_1^2) = 1 - \hat{\phi}_1^2 \neq 0$. Hence we reached contradiction. It follows that we can choose ϕ_1 and ϕ_2 so that all conditions (25) are satisfied. \square

6.3. A strong connection. Method I. In this subsection we construct a strong connection on the $C(\mathbb{Z}_2)$ -comodule algebra $C(S_{\mathbb{R}\mathcal{T}}^2)$ by repeated application of the formula stated in the proof of [13, Lemma 3.2]. Let P be a fibre product of $P_1 \xrightarrow{\pi_2^1} P_{12} \xleftarrow{\pi_2^1} P_2$ in the category of right H -comodule algebras. Assume that the maps π_j^i are surjective and that $\ell_i : H \rightarrow P_i \otimes P_i$, $i = 1, 2$, are strong connections. Then the formula ([13, Lemma 3.2])

$$(33) \quad \begin{aligned} \ell(h) &= \left(\ell_1(h)^{\langle 1 \rangle}, f_2^1(\ell_1(h)^{\langle 1 \rangle}) \right) \otimes \left(\ell_1(h)^{\langle 2 \rangle}, f_2^1(\ell_1(h)^{\langle 2 \rangle}) \right) \\ &+ \left(0, (\varepsilon(h_{(1)}) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle})f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})) \ell_2(h_{(2)})^{\langle 1 \rangle} \right) \otimes \left(f_1^2(\ell_2(h_{(2)})^{\langle 2 \rangle}), \ell_2(h_{(2)})^{\langle 2 \rangle} \right) \end{aligned}$$

defines a strong connection $\ell : H \rightarrow P \otimes P$. Here $f_j^i := \mu_j^i \circ \pi_j^i$ and μ_j^i is any unital colinear splitting of π_j^i , $i \neq j$. Note also that we use the convention that, if $x^{\langle 1 \rangle} \otimes x^{\langle 2 \rangle} := \sum_i x_i \otimes y_i$, then $(x^{\langle 1 \rangle}, x^{\langle 1 \rangle}) \otimes (x^{\langle 2 \rangle}, x^{\langle 2 \rangle}) := \sum_i (x_i, x_i) \otimes (y_i, y_i)$, and similarly for coproducts. Observe that for $C(\mathbb{Z}_2)$ -comodule algebras it is enough to compute the value of a strong connection for $h = u$, where u is the group-like generator of $C(\mathbb{Z}_2)$ because strong connections are unital and linear, i.e., it is sufficient to use the following equation:

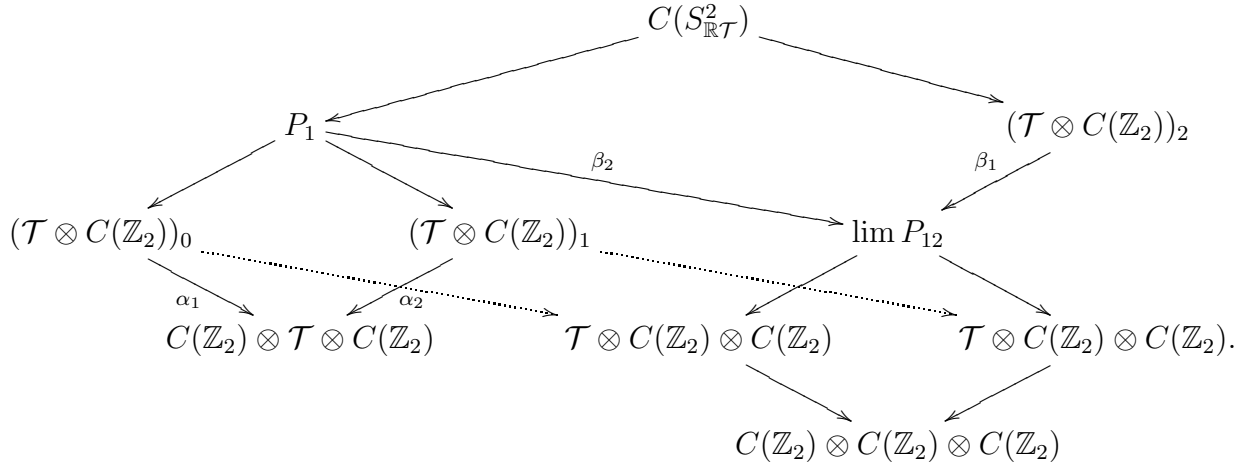
$$(34) \quad \begin{aligned} \ell(u) &= \left(\ell_1(u)^{\langle 1 \rangle}, f_2^1(\ell_1(u)^{\langle 1 \rangle}) \right) \otimes \left(\ell_1(u)^{\langle 2 \rangle}, f_2^1(\ell_1(u)^{\langle 2 \rangle}) \right) \\ &+ \left(0, (1 - f_2^1(\ell_1(u)^{\langle 1 \rangle})f_2^1(\ell_1(u)^{\langle 2 \rangle})) \ell_2(u)^{\langle 1 \rangle} \right) \otimes \left(f_1^2(\ell_2(u)^{\langle 2 \rangle}), \ell_2(u)^{\langle 2 \rangle} \right) \end{aligned}$$

Note that it is sufficient to know the values $f_j^i(x)$ only for a set of elements $x \in P_j$ which actually appear in the above formula and which (because of bi-colinearity of strong connections) can be assumed to be linearly independent and satisfy $\rho(x) = x \otimes u$, i.e., one needs only to solve the following equations with unknowns $f_j^i(x) \in P_j$ (where ρ denotes the coaction):

$$(35) \quad \rho(f_j^i(x)) = f_j^i(x) \otimes u, \quad \pi_i^j(f_j^i(x)) = \pi_j^i(x).$$

As the formula (33) assumes the comodule algebra to be presented as the ordinary (double) pullback, we need to convert the triple-pullback defining $C(S_{\mathbb{R}\mathcal{T}}^2)$ to an iterated pullback and apply the formula recursively. Since all the maps $C(\mathbb{R}P_{\mathcal{T}}^2) \rightarrow \mathcal{T}_i \otimes C(\mathbb{Z}_2)$ are surjective [18], we can apply [21, Lemma 0.2 and Proposition 1.3] to present $C(S_{\mathbb{R}\mathcal{T}}^2)$ as a desired iterated

pullback.
(36)



Here

$$(37) \quad \begin{aligned} \alpha_1 &= \sigma_1 \otimes \text{id}, & \beta_1(x) &= ((\Phi_{02} \circ (\sigma_1 \otimes \text{id}))(x), (\Phi_{12} \circ (\sigma_2 \otimes \text{id}))(x)), \\ \alpha_2 &= \Phi_{01} \circ (\sigma_1 \otimes \text{id}), & \beta_2(x, y) &= ((\sigma_2 \otimes \text{id})(x), (\sigma_2 \otimes \text{id})(y)). \end{aligned}$$

We will first compute a strong connection $\ell_{01} : C(\mathbb{Z}_2) \rightarrow P_1 \otimes P_1$ on P_1 – the fiber product of $\mathcal{T}_0 \otimes C(\mathbb{Z}_2)$ and $\mathcal{T}_1 \otimes C(\mathbb{Z}_2)$ (see (36)). We use the particular choice of the the strong connections ℓ_0 and ℓ_1 on trivial pieces $\mathcal{T}_0 \otimes C(\mathbb{Z}_2)$ and $\mathcal{T}_1 \otimes C(\mathbb{Z}_2)$ given by

$$(38) \quad \ell_0(u) = (1 \otimes u) \otimes (1 \otimes u), \quad \ell_1(u) = (1 \otimes u) \otimes (1 \otimes u).$$

Substituting the above formulae in (34) yields a glued strong connection on P_1 :

$$(39) \quad \begin{aligned} \ell_{01}(u) &= (1 \otimes u, f_1^0(1 \otimes u)) \otimes (1 \otimes u, f_1^0(1 \otimes u)) \\ &+ (0, (1 \otimes 1 - f_1^0(1 \otimes u)f_1^1(1 \otimes u))(1 \otimes u)) \otimes (f_1^1(1 \otimes u), 1 \otimes u) \end{aligned}$$

Let us write for brevity $a := f_1^0(1 \otimes u)$, $b := f_1^1(1 \otimes u)$. By Diagram(36) and Eq. (35) elements $a, b \in \mathcal{T} \otimes C(\mathbb{Z}_2)$ are any solutions to the following equations:

$$(40) \quad \begin{aligned} \rho(a) &= a \otimes u, & (\sigma_1 \otimes \text{id})(1 \otimes u) &= (\Phi_{01} \circ (\sigma_1 \otimes \text{id}))(a), \\ \rho(b) &= b \otimes u, & (\sigma_1 \otimes \text{id})(b) &= (\Phi_{01} \circ (\sigma_1 \otimes \text{id}))(1 \otimes u). \end{aligned}$$

Substituting the definition of Φ_{01} simplifies the above system of equations to

$$(41) \quad \rho(a) = a \otimes u, \quad \rho(b) = b \otimes u, \quad (\sigma_1 \otimes \text{id})(a) = u \otimes 1 \otimes 1, \quad (\sigma_1 \otimes \text{id})(b) = u \otimes 1 \otimes 1,$$

and it is easy to see that one of the solutions is

$$(42) \quad a = \phi_1 \otimes 1, \quad b = \phi_1 \otimes 1.$$

Here and in what follows ϕ_1 and ϕ_2 are elements of \mathcal{T} satisfying all the conditions (25). Substituting the above solution into (39) yields the following strong connection on P_1 :

$$(43) \quad \ell_{01}(u) = (1 \otimes u, \phi_1 \otimes 1) \otimes (1 \otimes u, \phi_1 \otimes 1) + (0, (1 - \phi_1^2) \otimes u) \otimes (\phi_1 \otimes 1, 1 \otimes u)$$

Now, we apply the formula (33) to the second iterated pullback in the diagram (36):

$$(44) \quad \begin{array}{ccc} \mathcal{T}_2 \otimes C(\mathbb{Z}_2) & & P_1 \\ & \searrow \beta_1 & \swarrow \beta_2 \\ & Q & \end{array} \quad \begin{aligned} \beta_1(x) &= ((\Phi_{02} \circ (\sigma_1 \otimes \text{id}))(x), (\Phi_{12} \circ (\sigma_2 \otimes \text{id}))(x)), \\ \beta_2(x, y) &= ((\sigma_2 \otimes \text{id})(x), (\sigma_2 \otimes \text{id})(y)). \end{aligned}$$

where $Q \subseteq (C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2)) \oplus (C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2))$. We choose the strong connection on P_1 given by eq. (43), and on $\mathcal{T}_2 \otimes C(\mathbb{Z}_2)$ given by

$$(45) \quad \ell_2(u) = (1 \otimes u) \otimes (1 \otimes u)$$

Substituting these into formula (34) yields a strong connection on $C(S_{\mathbb{R}\mathcal{T}}^2)$:

$$(46) \quad \begin{aligned} \ell(u) &= (\ell_2(u)^{\langle 1 \rangle}, f_{01}^2(\ell_2(u)^{\langle 1 \rangle})) \otimes (\ell_2(u)^{\langle 2 \rangle}, f_{01}^2(\ell_2(u)^{\langle 2 \rangle})) \\ &\quad + \left(0, ((1 \otimes 1, 1 \otimes 1) - f_{01}^2(\ell_2(u)^{\langle 1 \rangle}) f_{01}^2(\ell_2(u)^{\langle 2 \rangle})) \ell_{01}(u)^{\langle 1 \rangle} \right) \\ &\quad \otimes (f_2^{01}(\ell_{01}(u)^{\langle 2 \rangle}), \ell_{01}(u)^{\langle 2 \rangle}) \\ &= (1 \otimes u, f_{01}^2(1 \otimes u)) \otimes (1 \otimes u, f_{01}^2(1 \otimes u)) \\ &\quad + \left(0, ((1 \otimes 1, 1 \otimes 1) - (f_{01}^2(1 \otimes u))^2) (1 \otimes u, \phi_1 \otimes 1) \right) \\ &\quad \otimes (f_2^{01}((1 \otimes u, \phi_1 \otimes 1)), (1 \otimes u, \phi_1 \otimes 1)) \\ &\quad + \left(0, ((1 \otimes 1, 1 \otimes 1) - (f_{01}^2(1 \otimes u))^2) (0, (1 - \phi_1^2) \otimes u) \right) \\ &\quad \otimes (f_2^{01}((\phi_1 \otimes 1, 1 \otimes u)), (\phi_1 \otimes 1, 1 \otimes u)), \end{aligned}$$

where $f_{01}^2 : \mathcal{T} \otimes C(\mathbb{Z}_2) \rightarrow P_1$, $f_2^{01} : P_1 \rightarrow \mathcal{T} \otimes C(\mathbb{Z}_2)$ are any linear, unital, right $C(\mathbb{Z}_2)$ -comodule maps satisfying $\beta_2 \circ f_{01}^2 = \beta_1$, $\beta_1 \circ f_2^{01} = \beta_2$. Denote for brevity $(a_0, a_1) := f_{01}^2(1 \otimes u)$, $b := f_2^{01}((1 \otimes u, \phi_1 \otimes 1))$, $c := f_2^{01}((\phi_1 \otimes 1, 1 \otimes u))$. It follows that we need to solve the following system of equations for a_0 , a_1 , b , c :

$$(47) \quad \begin{aligned} \rho(a_0) &= a_0 \otimes u, & (\Phi_{02} \circ (\sigma_1 \otimes \text{id}))(1 \otimes u) &= (\sigma_2 \otimes \text{id})(a_0), \\ \rho(a_1) &= a_1 \otimes u, & (\Phi_{12} \circ (\sigma_2 \otimes \text{id}))(1 \otimes u) &= (\sigma_2 \otimes \text{id})(a_1), \\ \rho(b) &= b \otimes u, & (\Phi_{02} \circ (\sigma_1 \otimes \text{id}))(b) &= \sigma_2(1) \otimes u, \\ & & (\Phi_{12} \circ (\sigma_2 \otimes \text{id}))(b) &= \sigma_2(\phi_1) \otimes 1_{C(\mathbb{Z}_2)}, \\ \rho(c) &= c \otimes u, & (\Phi_{02} \circ (\sigma_1 \otimes \text{id}))(c) &= \sigma_2(\phi_1) \otimes 1_{C(\mathbb{Z}_2)}, \\ & & (\Phi_{12} \circ (\sigma_2 \otimes \text{id}))(c) &= \sigma_2(1) \otimes u. \end{aligned}$$

Simplification of the right column of the above equations using eq. (25) yields:

$$(48) \quad \begin{aligned} (\sigma_2 \otimes \text{id})(a_0) &= 1_{C(I)} \otimes u \otimes 1_{C(\mathbb{Z}_2)}, & (\sigma_2 \otimes \text{id})(a_1) &= 1_{C(I)} \otimes u \otimes 1_{C(\mathbb{Z}_2)}, \\ (\sigma_1 \otimes \text{id})(b) &= u \otimes 1_{C(I)} \otimes 1_{C(\mathbb{Z}_2)}, & (\sigma_2 \otimes \text{id})(b) &= \iota_I \otimes 1_{C(\mathbb{Z}_2)} \otimes 1_{C(\mathbb{Z}_2)}, \\ (\sigma_1 \otimes \text{id})(c) &= 1_{C(\mathbb{Z}_2)} \otimes \iota_I \otimes 1_{C(\mathbb{Z}_2)}, & (\sigma_2 \otimes \text{id})(c) &= 1_{C(I)} \otimes u \otimes 1_{C(\mathbb{Z}_2)}. \end{aligned}$$

Using eq. (28) again, one easily verifies that one of the solutions can be given as

$$(49) \quad (a_0, a_1) = (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}), \quad b = \phi_1 \otimes 1_{C(\mathbb{Z}_2)}, \quad c = \phi_2 \otimes 1_{C(\mathbb{Z}_2)}.$$

Substituting this particular solution to the formula (46) for $\ell(u)$ yields:

$$(50) \quad \begin{aligned} \ell(u) &= (1 \otimes u, (\phi_2 \otimes 1, \phi_2 \otimes 1)) \otimes (1 \otimes u, (\phi_2 \otimes 1, \phi_2 \otimes 1)) \\ &\quad + \left(0, ((1 \otimes 1, 1 \otimes 1) - ((\phi_2 \otimes 1, \phi_2 \otimes 1))^2) (1 \otimes u, \phi_1 \otimes 1) \right) \\ &\quad \otimes (\phi_1 \otimes 1, (1 \otimes u, \phi_1 \otimes 1)) \\ &\quad + \left(0, ((1 \otimes 1, 1 \otimes 1) - ((\phi_2 \otimes 1, \phi_2 \otimes 1))^2) (0, (1 - \phi_1^2) \otimes u) \right) \\ &\quad \otimes (\phi_2 \otimes 1, (\phi_1 \otimes 1, 1 \otimes u)). \end{aligned}$$

Simplifying, removing unnecessary parentheses and rearranging terms so that $C(S_{\mathbb{R}\mathcal{T}}^2) \subseteq (\mathcal{T}_0 \otimes C(\mathbb{Z}_2)) \oplus (\mathcal{T}_1 \otimes C(\mathbb{Z}_2)) \oplus (\mathcal{T}_2 \otimes C(\mathbb{Z}_2))$ yields finally:

$$(51) \quad \begin{aligned} \ell(u) = & (\phi_2 \otimes 1, \phi_2 \otimes 1, 1 \otimes u) \otimes (\phi_2 \otimes 1, \phi_2 \otimes 1, 1 \otimes u) \\ & + ((1 - \phi_2^2) \otimes u, (1 - \phi_2^2)\phi_1 \otimes 1, 0) \otimes (1 \otimes u, \phi_1 \otimes 1, \phi_1 \otimes 1) \\ & + (0, (1 - \phi_2^2)(1 - \phi_1^2) \otimes u, 0) \otimes (\phi_1 \otimes 1, 1 \otimes u, \phi_2 \otimes 1). \end{aligned}$$

Write $\ell(u) = \sum_{i=1}^3 l_i \otimes r_i$ where

$$(52) \quad \begin{aligned} l_1 &= (\phi_2 \otimes 1, \phi_2 \otimes 1, 1 \otimes u), & r_1 &= (\phi_2 \otimes 1, \phi_2 \otimes 1, 1 \otimes u), \\ l_2 &= ((1 - \phi_2^2) \otimes u, (1 - \phi_2^2)\phi_1 \otimes 1, 0), & r_2 &= (1 \otimes u, \phi_1 \otimes 1, \phi_1 \otimes 1), \\ l_3 &= (0, (1 - \phi_2^2)(1 - \phi_1^2) \otimes u, 0), & r_3 &= (\phi_1 \otimes 1, 1 \otimes u, \phi_2 \otimes 1). \end{aligned}$$

According to [5, Theorem 3.1] if both $\{l_1, l_2, l_3\}$ and $\{r_1, r_2, r_3\}$ are (separately) sets of linearly independent vectors then

$$p_{ij} := r_i l_j, \quad p := (p_{ij}) \in M_3(C(\mathbb{R}P_{\mathcal{T}}^2)), \quad p^2 = p,$$

is a projector for an associated line bundle. Hence, in order to use this result, we need to prove that zeros are the only solutions to equations

$$(53) \quad \sum_{i=1}^3 \alpha_i r_i = 0, \quad \sum_{i=1}^3 \beta_i l_i = 0$$

The first of the above equalities implies that $\alpha_1 \phi_2 \otimes 1 + \alpha_2 1 \otimes u + \alpha_3 \phi_1 \otimes 1 = 0$, hence immediately $\alpha_2 = 0$. Because $\{\hat{\phi}_1, \hat{\phi}_2\} = \sigma(\{\phi_1, \phi_2\})$ (eq. (25)) are linearly independent in $C(S^1)$, which can be checked easily by direct computation using eq. (26), also $\{\phi_1, \phi_2\}$ must be linearly independent in \mathcal{T} , hence $\alpha_1 = \alpha_3 = 0$. The second equality in eq. (53) can be expanded as:

$$(54) \quad \beta_1 \phi_2 \otimes 1 + \beta_2 (1 - \phi_2^2) \otimes u = 0,$$

$$(55) \quad \beta_1 \phi_2 \otimes 1 + \beta_2 (1 - \phi_2^2) \phi_1 \otimes 1 + \beta_3 (1 - \phi_2^2)(1 - \phi_1^2) \otimes u = 0,$$

$$(56) \quad \beta_1 1 \otimes u = 0.$$

It follows immediately from eq. (56) that $\beta_1 = 0$. Then because $\sigma(1 - \phi_2^2) = 1 - \hat{\phi}_2^2 \neq 0$ (see eq. (25),(26)) we have also $1 - \phi_2^2 \neq 0$, and so, by eq. (54), $\beta_2 = 0$. Finally, Equation (55) and Equation (25c) implies that $\beta_3 = 0$.

6.4. A strong connection. Method II. In this subsection we construct a strong connection on the $C(\mathbb{Z}_2)$ -comodule algebra $C(S_{\mathbb{R}\mathcal{T}}^2)$ using the formula given in Theorem 6. As in the previous subsection $\phi_1, \phi_2 \in \mathcal{T}$ denote some chosen elements satisfying all the conditions in (25). Also as before, the strong connections on the three copies of $C(\mathbb{Z}_2)$ -comodule algebra (with diagonal coaction) $\mathcal{T} \otimes C(\mathbb{Z}_2)$ which are components of $C(S_{\mathbb{R}\mathcal{T}}^2)$ are chosen as given by the formulas

$$\ell_1(u) = \ell_2(u) = \ell_3(u) = 1_{\mathcal{T}} \otimes u, \quad \ell_1(1_{C(\mathbb{Z}_2)}) = \ell_2(1_{C(\mathbb{Z}_2)}) = \ell_3(1_{C(\mathbb{Z}_2)}) = 1_{\mathcal{T}} \otimes 1_{C(\mathbb{Z}_2)}.$$

In order to use the formula from Theorem 6 we need the appropriate colinear and unital splittings from the linear subspaces generated by the legs of ℓ_i 's into $C(S_{\mathbb{R}\mathcal{T}}^2)$. The reader will easily verify that the maps $\alpha_i : \text{Span}\{1_{\mathcal{T}} \otimes u, 1_{\mathcal{T}} \otimes 1_{C(\mathbb{Z}_2)}\} \rightarrow C(S_{\mathbb{R}\mathcal{T}}^2)$, $i = 0, 1, 2$ defined by

$\alpha_i(1_{\mathcal{T}} \otimes 1_{C(\mathbb{Z}_2)}) := 1_{C(S^2_{\mathbb{R}\mathcal{T}})}$, for $i = 0, 1, 2$, and

$$\begin{aligned} \alpha_0(1_{\mathcal{T}} \otimes u) &:= (1_{\mathcal{T}} \otimes u, \phi_1 \otimes 1_{C(\mathbb{Z}_2)}, \phi_1 \otimes 1_{C(\mathbb{Z}_2)}), \\ \alpha_1(1_{\mathcal{T}} \otimes u) &:= (\phi_1 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}), \\ (57) \quad \alpha_2(1_{\mathcal{T}} \otimes u) &:= (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u). \end{aligned}$$

Incidentally, the above formulas were not guessed but derived using the degenerated version of the construction from Theorem 7 in which the relevant parts of α_j^i 's were obtained utilizing ϕ_1 and ϕ_2 . By luck, the corrections in which the splittings β_j^i could have been used turned out to be unnecessary — hence it was also unnecessary to derive β_j^i 's using the methods from the Section 5.

Let us denote for brevity $\gamma_i := \alpha_i(1_{\mathcal{T}} \otimes u)$, as well as omit subscripts indicating the algebra the unit elements belong to. Let us note that because $u^2 = 1$ we have

$$1 - \alpha_1^2 = ((1 - \phi_1^2) \otimes 1, 0, (1 - \phi_2^2) \otimes 1), \quad 1 - \alpha_1^2 = ((1 - \phi_2^2) \otimes 1, (1 - \phi_2^2) \otimes 1, 0).$$

Then the straightforward application of the formula from the Theorem 6 yields:

$$\begin{aligned} \ell(u) &:= \alpha_0 \otimes \alpha_0(1 - \alpha_1^2)(1 - \alpha_2^2) + \alpha_1 \otimes \alpha_1(1 - \alpha_2^2) + \alpha_2 \otimes \alpha_2 \\ &= (1 \otimes u, \phi_1 \otimes 1, \phi_1 \otimes 1) \otimes ((1 - \phi_1^2)(1 - \phi_2^2) \otimes u, 0, 0) \\ &\quad + (\phi_1 \otimes 1, 1 \otimes u, \phi_2 \otimes 1) \otimes (\phi_1(1 - \phi_2^2) \otimes 1, (1 - \phi_2^2) \otimes u, 0) \\ (58) \quad &\quad + (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u) \otimes (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u). \end{aligned}$$

Note the similarity of this formula to the formula (51) obtained using the other method in the previous subsection. This similarity is understandable, because (not excluding the possibility of some general link, as yet unexplored by the author, between the two methods used) the common feature of both particular computations is that by construction, both strong connection formulas were expressed using the limited set of elements: $\phi_1, \phi_2, 1_{\mathcal{T}} \in \mathcal{T}$, and $1_{C(\mathbb{Z}_2)}, u \in C(\mathbb{Z}_2)$.

We leave to the reader analogous computations as those at the end of the previous subsection, which prove that both left and right legs of the above strong connection are linearly independent (when taken separately).

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